Math 181 Final Review Study Guide
Problem 1

1. Differentiate the function:

$$F(x) = \int_{\sqrt{x}}^{x^2} e^{t^3}\, dt$$

Solution: Using the Fundamental Theorem of Calculus Part II and the Chain Rule, the derivative of $F(x) = \displaystyle\int_{g(x)}^{h(x)} f(t)\, dt$ is:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\, dt \\ &= f(h(x)) \cdot \frac{d}{dx}h(x) - f(g(x)) \cdot \frac{d}{dx}g(x) \end{aligned}$$

Applying the formula to the given function $F(x)$ we get:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_{\sqrt{x}}^{x^2} e^{t^3}\, dt \\ &= e^{(x^2)^3} \cdot \frac{d}{dx}\left(x^2\right) - e^{(\sqrt{x})^3} \cdot \frac{d}{dx}\left(\sqrt{x}\right) \\ &= \boxed{e^{x^6} \cdot (2x) - e^{x^{3/2}} \cdot \left(\frac{1}{2\sqrt{x}}\right)} \end{aligned}$$

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Problem 2

2. Compute the definite integral:

$$\int_0^1 xe^{3x}\,dx$$

Solution: We will evaluate the integral using Integration by Parts. Let $u = x$ and $v' = e^{3x}$. Then $u' = 1$ and $v = \frac{1}{3}e^{3x}$. Using the Integration by Parts formula:

$$\int_a^b uv'\,dx = \Big[uv\Big]_a^b - \int_a^b u'v\,dx$$

we get:

$$\begin{aligned}
\int_0^1 xe^{3x}\,dx &= \left[\frac{1}{3}xe^{3x}\right]_0^1 - \int_0^1 \frac{1}{3}e^{3x}\,dx \\
&= \left[\frac{1}{3}xe^{3x}\right]_0^1 - \left[\frac{1}{9}e^{3x}\right]_0^1 \\
&= \left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right]_0^1 \\
&= \left[\frac{1}{3}(1)e^{3(1)} - \frac{1}{9}e^{3(1)}\right] - \left[\frac{1}{3}(0)e^{3(0)} - \frac{1}{9}e^{3(0)}\right] \\
&= \boxed{\frac{2}{9}e^3 + \frac{1}{9}}
\end{aligned}$$

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Problem 3

1. Compute the integrals:

$$\int \frac{x}{\sqrt{x-2}}\,dx \qquad \int x^3\sin(x^2)\,dx \qquad \int \frac{dx}{x^2+x-6}\,dx$$

$$\int \frac{dx}{x^2+x+3} \qquad \int \frac{dx}{x^3-x} \qquad \int x^6\ln x\,dx$$

$$\int \arctan x\,dx \qquad \int \cos(\sqrt{x})\,dx \qquad \int x^2e^{2x}\,dx$$

Solution:

(1) The first integral is computed using the u-substitution method. Let $u = x - 2$. Then $du = dx$ and $x = u + 2$. Substituting these into the integral and evaluating we get:

$$\begin{aligned}\int \frac{x}{\sqrt{x-2}}\,dx &= \int \frac{u+2}{\sqrt{u}}\,du \\ &= \int \left(u^{1/2} + 2u^{-1/2}\right)\,du \\ &= \frac{2}{3}u^{3/2} + 4u^{1/2} + C \\ &= \boxed{\frac{2}{3}(x-2)^{3/2} + 4(x-2)^{1/2} + C}\end{aligned}$$

(2) The second integral is computed using the u-substitution method. Let $u = x^2$. Then $du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}\,du = x\,dx$ and we get:

$$\begin{aligned}\int x^3\sin\left(x^2\right)\,dx &= \int x^2\sin\left(x^2\right)\,x\,dx \\ &= \int u\sin u\left(\frac{1}{2}\,du\right) \\ &= \frac{1}{2}\int u\sin u\,du\end{aligned}$$

We now use Integration by Parts to evaluate the above integral. Let $w = u$ and $v' = \sin u$. Then $w' = 1$ and $v = -\cos u$. Using the Integration by Parts formula:

$$\int wv'\,du = wv - \int w'v\,du$$

we get:

$$\begin{aligned}\int u\sin u\,du &= u(-\cos u) - \int 1\cdot(-\cos u)\,du \\ &= -u\cos u + \int \cos u\,du \\ &= -u\cos u + \sin u + C\end{aligned}$$

Therefore,

$$\begin{aligned}\int x^3\sin\left(x^2\right)\,dx &= \frac{1}{2}\int u\sin u\,du \\ &= \frac{1}{2}\left(-u\cos u + \sin u\right) + C \\ &= -\frac{1}{2}u\cos u + \frac{1}{2}\sin u + C \\ &= \boxed{-\frac{1}{2}x^2\cos\left(x^2\right) + \frac{1}{2}\sin\left(x^2\right) + C}\end{aligned}$$

(3) The third integral is computed using Partial Fraction Decomposition. Factoring the denominator and decomposing we get:

$$\frac{1}{x^2+x-6} = \frac{1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Multiplying the equation by $(x+3)(x-2)$ we get:

$$1 = A(x-2) + B(x+3)$$

Next we plug in two different values of x to get a system of two equations in two unknowns (A, B). Letting $x=-3$ and $x=2$ we get:

$$\begin{aligned}x=-3: \quad 1 = A(-3-2) + B(-3+3) \quad &\Rightarrow \quad A = -\frac{1}{5} \\ x=2: \quad 1 = A(2-2) + B(2+3) \quad &\Rightarrow \quad B = \frac{1}{5}\end{aligned}$$

Plugging these values of A and B back into the decomposed equation and integrating we get:

$$\begin{aligned}\int \frac{1}{x^2+x-6}\,dx &= \int \left(\frac{-\frac{1}{5}}{x+3} + \frac{\frac{1}{5}}{x-2}\right)dx \\ &= \boxed{-\frac{1}{5}\ln|x+3| + \frac{1}{5}\ln|x-2| + C}\end{aligned}$$

(4) We begin solving the fourth integral by completing the square in the denominator.

$$\int \frac{1}{x^2+x+3}\,dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{11}{4}}\,dx$$

We then evaluate the integral using the u-substitution method. Let $\frac{\sqrt{11}}{2}u = x+\frac{1}{2}$. Then $\frac{\sqrt{11}}{2}\,du = dx$ and we get:

$$\begin{aligned}
\int \frac{1}{x^2+x+3}\,dx &= \int \frac{1}{(x+\frac{1}{2})^2+\frac{11}{4}}\,dx \\
&= \frac{\sqrt{11}}{2}\int \frac{1}{\frac{11}{4}u^2+\frac{11}{4}}\,du \\
&= \frac{2}{\sqrt{11}}\int \frac{1}{u^2+1}\,du \\
&= \frac{2}{\sqrt{11}}\arctan u + C \\
&= \boxed{\frac{2}{\sqrt{11}}\arctan\left[\frac{2}{\sqrt{11}}\left(x+\frac{1}{2}\right)\right]+C}
\end{aligned}$$

(5) We will evaluate the fifth integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{x^3-x} = \frac{1}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}$$

Next, we multiply the above equation by $x(x-1)(x+1)$ to get:

$$1 = A(x-1)(x+1)+Bx(x+1)+Cx(x-1)$$

Then we plug in three different values for x to create a system of three equations in three unknowns (A, B, C). We select $x=0$, $x=-1$, and $x=1$ for simplicity.

$$\begin{aligned}
x=0:&\ \ A(0-1)(0+1)=1 \ \Rightarrow\ -A=1 \ \Rightarrow\ A=-1 \\
x=-1:&\ \ C(-1)(-1-1)=1 \ \Rightarrow\ 2C=1 \ \Rightarrow\ C=\frac{1}{2} \\
x=1:&\ \ B(1)(1+1)=1 \ \Rightarrow\ 2B=1 \ \Rightarrow\ B=\frac{1}{2}
\end{aligned}$$

Finally, we plug these values for A, B, and C back into the decomposition and integrate.

$$\begin{aligned}\int \frac{dx}{x^3-4x} &= \int \left(\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}\right)dx \\ &= \int \left(\frac{-1}{x}+\frac{\frac{1}{2}}{x-1}+\frac{\frac{1}{2}}{x+1}\right)dx \\ &= \boxed{-\ln|x|+\frac{1}{2}\ln|x-1|+\frac{1}{2}\ln|x+1|+C}\end{aligned}$$

(6) The sixth integral is computed using Integration by Parts. Let $u=\ln x$ and $v'=x^6$. Then $u'=\frac{1}{x}$ and $v=\frac{1}{7}x^7$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}\int x^6 \ln x\,dx &= (\ln x)\left(\frac{1}{7}x^7\right) - \int \frac{1}{x}\cdot\frac{1}{7}x^7\,dx \\ &= \frac{1}{7}x^7\ln x - \frac{1}{7}\int x^6\,dx \\ &= \boxed{\frac{1}{7}x^7\ln x - \frac{1}{49}x^7 + C}\end{aligned}$$

(7) We will evaluate the seventh integral using Integration by Parts. Let $u=\arctan x$ and $v'=1$. Then $u'=\frac{1}{x^2+1}$ and $v=x$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\int \arctan x\,dx = x\arctan x - \int \frac{1}{x^2+1}\,x\,dx.$$

Use the substitution $w=x^2+1$ to evaluate the integral on the right hand side. Then $dw = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}dw = x\,dx$ and we get:

$$\begin{aligned}\int \arctan x\,dx &= x\arctan x - \frac{1}{2}\int \frac{1}{w}\,dw \\ &= x\arctan x - \frac{1}{2}\ln|w| + C \\ &= \boxed{x\arctan x - \frac{1}{2}\ln(x^2+1)+C}\end{aligned}$$

Note that the absolute value signs aren't needed because $x^2+1>0$ for all x.

(8) To begin the solution of the eighth integral, we first use the u-substitution method. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx \quad \Rightarrow \quad 2u\,du = dx$ and we get:

$$\begin{aligned}\int \cos\left(\sqrt{x}\right)\,dx &= \int \cos u\,(2u\,du)\\ &= 2\int u\cos u\,du\end{aligned}$$

We now use Integration by Parts to evaluate the above integral. Let $w = u$ and $v' = \cos u$. Then $w' = 1$ and $v = \sin u$. Using the Integration by Parts formula:

$$\int wv'\,du = wv - \int w'v\,du$$

we get:

$$\begin{aligned}\int u\cos u\,du &= u\sin u - \int \sin u\,du\\ \int u\cos u\,du &= u\sin u + \cos u + C\end{aligned}$$

Therefore,

$$\begin{aligned}\int \cos\left(\sqrt{x}\right)\,dx &= 2\int u\cos u\,du\\ &= 2\left(u\sin u + \cos u\right) + C\\ &= 2u\sin u + \frac{1}{2}\cos u + C\\ &= \boxed{2\sqrt{x}\sin\left(\sqrt{x}\right) + 2\cos\left(\sqrt{x}\right) + C}\end{aligned}$$

(9) The last integral is computed using Integration by Parts. Let $u = x^2$ and $v' = e^{2x}$. Then $u' = 2x$ and $v = \dfrac{1}{2}e^{2x}$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}\int x^2e^{2x}\,dx &= \frac{1}{2}x^2e^{2x} - \int 2x\left(\frac{1}{2}e^{2x}\right)\,dx\\ &= \frac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx\end{aligned}$$

A second Integration by Parts must be performed. Let $u = x$ and $v' = e^{2x}$. Then $u' = 1$ and $v = \frac{1}{2}e^{2x}$. Using the Integration by Parts formula again we get:

$$\begin{aligned}\int x^2 e^{2x}\,dx &= \frac{1}{2}x^2e^{2x} - \left[\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx\right] \\ &= \boxed{\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C}\end{aligned}$$

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Problem 4

4. Determine if the following improper integrals converge or not.

$$\int_0^{+\infty} xe^{-2x}\,dx \qquad \int_0^{+\infty} \frac{dx}{x^2+4} \qquad \int_0^1 \frac{e^x}{\sqrt{x}}\,dx \qquad \int_1^{+\infty} \frac{x^{3/2}+3}{\sqrt{x}}\,dx$$

Solution: We evaluate the first integral by turning it into a limit calculation.

$$\int_0^{+\infty} xe^{-2x}\,dx = \lim_{R\to+\infty} \int_0^R xe^{-2x}\,dx$$

We use Integration by Parts to compute the integral. Let $u = x$ and $v' = e^{-2x}$. Then $u' = 1$ and $v = -\frac{1}{2}e^{-2x}$. Using the Integration by Parts formula we get:

$$\begin{aligned}
\int_a^b uv'\,dx &= \Big[uv\Big]_a^b - \int_a^b u'v\,dx \\
\int_0^R xe^{-2x}\,dx &= \left[-\frac{1}{2}xe^{-2x}\right]_0^R - \int_0^R \left(-\frac{1}{2}e^{-2x}\right)\,dx \\
&= \left[-\frac{1}{2}xe^{-2x}\right]_0^R + \frac{1}{2}\int_0^R e^{-2x}\,dx \\
&= \left[-\frac{1}{2}xe^{-2x}\right]_0^R + \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_0^R \\
&= \left[-\frac{1}{2}Re^{-2R} + \frac{1}{2}(0)e^{-2(0)}\right] + \frac{1}{2}\left[-\frac{1}{2}e^{-2R} + \frac{1}{2}e^{-3(0)}\right] \\
&= -\frac{R}{2e^{2R}} - \frac{1}{4e^{2R}} + \frac{1}{4}
\end{aligned}$$

We now take the limit of the above function as $R \to +\infty$.

$$\begin{aligned}
\int_0^{+\infty} xe^{-2x}\,dx &= \lim_{R\to+\infty}\int_0^R xe^{-2x}\,dx \\
&= \lim_{R\to+\infty}\left(-\frac{R}{2e^{2R}} - \frac{1}{4e^{2R}} + \frac{1}{4}\right) \\
&= -0 - 0 + \frac{1}{4} \\
&= \boxed{\frac{1}{4}}
\end{aligned}$$

We evaluate the second integral as follows:

$$\begin{aligned}
\int_0^{+\infty} \frac{dx}{x^2+4} &= \lim_{R\to+\infty} \int_0^R \frac{dx}{x^2+4} \\
&= \lim_{R\to+\infty} \left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_0^R \\
&= \lim_{R\to+\infty} \left[\frac{1}{2}\arctan\left(\frac{R}{2}\right) - \frac{1}{2}\arctan\left(\frac{0}{2}\right)\right] \\
&= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}(0) \\
&= \boxed{\frac{\pi}{4}}
\end{aligned}$$

We will use the Comparison Test to show that the third integral converges. Let $g(x) = \frac{e^x}{\sqrt{x}}$. We must choose a function $f(x)$ that satisfies:

$$(1) \int_0^1 f(x)\,dx \text{ converges} \quad \text{and} \quad (2)\ \ 0 \le g(x) \le f(x) \text{ for } 0 \le x \le 1$$

We choose $f(x) = \frac{e}{\sqrt{x}}$. This function satisfies the inequality:

$$\begin{aligned}
0 &\le g(x) \le f(x) \\
0 &\le \frac{e^x}{\sqrt{x}} \le \frac{e}{\sqrt{x}}
\end{aligned}$$

for $0 \le x \le 1$ because the denominator of $g(x)$ is greater than the denominator of $f(x)$ for these values of x. Furthermore, the integral $\int_0^1 f(x)\,dx = \int_0^1 \frac{e}{\sqrt{x}}\,dx$ converges because it is a p-integral with $p = \frac{1}{2} < 1$. Therefore, the integral $\int_0^1 g(x)\,dx = \int_0^1 \frac{e^x}{\sqrt{x}}\,dx$ converges by the Comparison Test.

We split the fourth integral into two integrals.

$$\int_1^{+\infty} \frac{x^{3/2}+3}{\sqrt{x}}\,dx = \int_1^{+\infty} \frac{x^{3/2}}{\sqrt{x}}\,dx + \int_1^{+\infty} \frac{3}{\sqrt{x}}\,dx = \int_1^{+\infty} x\,dx + 3\int_1^{+\infty} \frac{dx}{\sqrt{x}}$$

Both integrals on the far right hand side of the above equation are divergent p-integrals ($p = -1$ and $p = \frac{1}{2}$, both of which are less than 1). Thus, the integral diverges.

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Problem 5

5. Let R be the region included by the curves $y = 0$ and $y = x^2 + x$ between $x = 0$ and $x = 1$. Find the volume of the solid of revolution when the axis is the line $y = -1$.

Solution: We find the volume using the Washer method. The formula we will use is:

$$V = \pi \int_a^b \left[(\text{top} - (-1))^2 - (\text{bottom} - (-1))^2\right] dx$$

where the top curve is $y = x + x^2$ and the bottom curve is $y = 0$. The lower limit is $a = 0$ and the upper limit is $b = 1$. The volume is then:

$$\begin{aligned}
V &= \pi \int_a^b \left[(\text{top} - (-1))^2 - (\text{bottom} - (-1))^2\right] dx \\
&= \pi \int_0^1 \left[\left(x + x^2 + 1\right)^2 - (0+1)^2\right] dx \\
&= \pi \int_0^1 \left(x^2 + x^3 + x + x^3 + x^4 + x^2 + x + x^2 + 1 - 1\right) dx \\
&= \pi \int_0^1 \left(x^4 + 2x^3 + 3x^2 + 2x\right) dx \\
&= \pi \left[\frac{x^5}{5} + \frac{x^4}{2} + x^3 + x^2\right]_0^1 \\
&= \pi \left[\frac{1}{5} + \frac{1}{2} + 1 + 1\right] \\
&= \boxed{\frac{27\pi}{10}}
\end{aligned}$$

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Problem 6

6. Find the arclength of the graph of the function $y = 2x^{3/2} + 5$ between $x = 0$ and $x = 1$.

Solution: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + (y')^2}\, dx \\ &= \int_0^1 \sqrt{1 + \left(3x^{1/2}\right)^2}\, dx \\ &= \int_0^1 \sqrt{1 + 9x}\, dx \end{aligned}$$

We now use the u-substitution $u = 1 + 9x$. Then $\frac{1}{9}\, du = dx$, the lower limit of integration changes from 0 to 1, and the upper limit of integration changes from 1 to 10.

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + 9x}\, dx \\ &= \frac{1}{9} \int_1^{10} \sqrt{u}\, du \\ &= \frac{1}{9} \left[\frac{2}{3} u^{3/2} \right]_1^{10} \\ &= \frac{1}{9} \left[\frac{2}{3} (10)^{3/2} - \frac{2}{3} (1)^{3/2} \right] \\ &= \boxed{\frac{2}{27} \left[10\sqrt{10} - 1 \right]} \end{aligned}$$

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Problem 7

7. Let $f(x) = x^2$ on the interval $[0, 1]$. Compute Mid(3) and Trap(3). Which one is an overestimate and why?

Solution: The length of each subinterval of $[0, 1]$ is:

$$\Delta x = \frac{b-a}{N} = \frac{1-0}{3} = \frac{1}{3}$$

The estimate Mid(3) is:

$$\begin{aligned}
\text{Mid}(3) &= \Delta x \left[f\left(\frac{1}{6}\right) + f\left(\frac{3}{6}\right) + f\left(\frac{5}{6}\right) \right] \\
&= \frac{1}{3}\left[\left(\frac{1}{6}\right)^2 + \left(\frac{3}{6}\right)^2 + \left(\frac{5}{6}\right)^2 \right] \\
&= \frac{1}{3}\left[\frac{1}{36} + \frac{9}{36} + \frac{25}{36} \right] \\
&= \boxed{\frac{35}{108}}
\end{aligned}$$

The estimate Trap(3) is:

$$\begin{aligned}
\text{Trap}(3) &= \frac{\Delta x}{2}\left[f(0) + 2f\left(\frac{1}{3}\right) + 2f\left(\frac{2}{3}\right) + f(1) \right] \\
&= \frac{\frac{1}{3}}{2}\left[0^2 + 2\left(\frac{1}{3}\right)^2 + 2\left(\frac{2}{3}\right)^2 + 1^2 \right] \\
&= \frac{1}{6}\left[0 + \frac{2}{9} + \frac{8}{9} + 1 \right] \\
&= \boxed{\frac{19}{54}}
\end{aligned}$$

Trap(3) is an overestimate because $f(x)$ is concave up on $[0, 1]$.

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Problem 8

9. Determine whether the following series converge or not:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{\sqrt{n^3+n+5}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^3+1)}{2^n}$ (c) $\displaystyle\sum_{n=1}^{+\infty} \frac{(n^2+1)3^n}{n!}$ (d) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{n \ln n}$

Solution:

(a) We note that:

$$0 \le \frac{n}{\sqrt{n^3+n^3+n^3}} \le \frac{n+2}{\sqrt{n^3+n+5}}$$
$$0 \le \frac{1}{\sqrt{3n}} \le \frac{n+2}{\sqrt{n^3+n+5}}$$

for $n \ge 1$ and that $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{3n}}$ is a divergent p-series with $p = \frac{1}{2} < 1$. Therefore, the series $\sum_{n=1}^{+\infty} \frac{n+2}{\sqrt{n^3+n+5}}$ **diverges** by the Comparison Test.

(b) This series is alternating so we can use the Leibniz Test. The function $\frac{a_n}{=} \frac{n^3+1}{2^n}$ is positive and decreasing for $n \ge 1$. Furthermore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3+1}{2^n} = 0$$

Therefore, the series **converges**.

(c) We will use the Ratio Test to show that the series converges.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \frac{(n+1)^2+1)3^{n+1}}{(n+1)!} \cdot \frac{n!}{(n^2+1)3^n} \\
&= \lim_{n\to\infty} \frac{n^2+2n+2}{n^2+1} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{n^2+2n+2}{n^2+1} \cdot 3 \cdot \frac{1}{n+1} \\
&= 3 \lim_{n\to\infty} \frac{n^2+2n+2}{n^3+n^2+n+1} \\
&= 0
\end{aligned}$$

Since $\rho = 0 < 1$, the series **converges**.

(d) For this series, we use the Integral Test. This is possible because the function $f(x) = \frac{1}{x \ln x}$ is positive and decreasing for $x \geq 2$. The integral of $f(x)$ on $[2, \infty)$ is:

$$\begin{aligned}
\int_2^{+\infty} \frac{dx}{x \ln x} &= \lim_{R \to +\infty} \int_2^R \frac{dx}{x \ln x} \\
&= \lim_{R \to +\infty} \Big[\ln(\ln x) \Big]_2^R \\
&= \lim_{R \to +\infty} \Big[\ln(\ln R) - \ln(\ln 2) \Big] \\
&= \infty
\end{aligned}$$

Thus, the series **diverges**.

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Problem 9

10. i) Show that the series

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n2^n}$$

converges. ii) Let $S(k)$ be defined by:

$$S(k) = \sum_{n=1}^{k} \frac{(-1)^n}{n2^n}$$

Then $S(k)$ is the partial sum of the series. Compute k so that $S(k)$ is within .01 of the sum of the series.

Solution: i) The series is alternating so we can use the Leibniz Test. The function $a_n = \frac{1}{n2^n}$ is positive and decreasing for $n \geq 1$. Furthermore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n2^n} = 0$$

Therefore, the series **converges**.

ii) Let S be the exact sum of the series. Then the error $|S - S(k)|$ is bounded as follows:

$$|S - S(k)| \leq a_{k+1}$$

We want the error to be smaller than .01, so we want to choose k to satisfy the following inequality:

$$|S - S(k)| \leq \frac{1}{(k+1)2^{k+1}} < \frac{1}{100}$$

Using a trial and error process, we find that if $k = 3$ then $\frac{1}{(3+1)2^{3+1}} = \frac{1}{64}$ and if $k = 4$ then $\frac{1}{(4+1)2^{4+1}} = \frac{1}{160}$. Thus, we choose $k \geq 4$.

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Problem 10

11. Compute the interval of convergence of the following power series:

$$\sum_{n=0}^{+\infty} \frac{(-2)^n(x+4)^n}{n+3} \qquad \sum_{n=0}^{+\infty} \frac{3^n(x-1)^{2n}}{n^2}$$

Solution: We use the Ratio Test to find the intervals of convergence. For the first series we have:

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(-2)^{n+1}(x+4)^{n+1}}{(n+1)+3} \cdot \frac{n+3}{2^n(x+4)^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(-2)^{n+1}}{(-2)^n} \cdot \frac{n+3}{n+4} \cdot \frac{(x+4)^{n+1}}{(x+4)^n}\right| \\
&= \lim_{n\to\infty} \left|-2 \cdot \frac{n+3}{n+4} \cdot (x+4)\right| \\
&= 2|x+4| \lim_{n\to\infty} \frac{n+3}{n+4} \\
&= 2|x+4| \cdot (1) \\
&= 2|x+4|
\end{aligned}$$

The series converges when $\rho = 2|x+4| < 1$ which gives us:

$$|x+4| < \frac{1}{2} \iff -\frac{1}{2} < x+4 < \frac{1}{2} \iff -\frac{9}{2} < x < -\frac{7}{2}$$

We must now check the endpoints. Plugging $x = -\frac{7}{2}$ into the given power series we get:

$$\sum_{n=0}^{+\infty} \frac{(-2)^n(-\frac{7}{2}+4)^n}{n+3} = \sum_{n=0}^{+\infty} \frac{(-2)^n(\frac{1}{2})^n}{n+3} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+3}$$

which converges by the Leibniz Test.

$$\sum_{n=0}^{+\infty} \frac{(-2)^n(-\frac{9}{2}+4)^n}{n+3} = \sum_{n=0}^{+\infty} \frac{(-2)^n(-\frac{1}{2})^n}{n+3} = \sum_{n=0}^{+\infty} \frac{1}{n+3}$$

which diverges by a direct comparison with $\sum_{n=1}^{+\infty} \frac{1}{n}$ which is a divergent p-series. Thus, the interval of convergence is:

$$\boxed{-\frac{9}{2} < x \leq -\frac{7}{2}}$$

For the second series we have:

$$
\begin{aligned}
\rho &= \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty} \left|\frac{3^{n+1}(x-1)^{2(n+1)}}{(n+1)^2} \cdot \frac{n^2}{3^n(x-1)^{2n}}\right| \\
&= \lim_{n\to\infty} \left|\frac{3^{n+1}}{3^n} \cdot \frac{n^2}{(n+1)^2} \cdot \frac{(x-1)^{2n+2}}{(x-1)^{2n}}\right| \\
&= \lim_{n\to\infty} \left|3 \cdot \left(\frac{n}{n+1}\right)^2 \cdot (x-1)^2\right| \\
&= 3|x-1|^2 \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^2 \\
&= 3|x-1|^2 \cdot (1)^2 \\
&= 3|x-1|^2
\end{aligned}
$$

The series converges when $\rho = 3|x-1|^2 < 1$ which gives us:

$$|x-1|^2 < \frac{1}{3} \iff |x-1| < \frac{1}{\sqrt{3}} \iff -\frac{1}{\sqrt{3}} + 1 < x < \frac{1}{\sqrt{3}} + 1$$

We must now check the endpoints. Plugging $x = \frac{1}{\sqrt{3}} + 1$ into the given power series we get:

$$\sum_{n=1}^{+\infty} \frac{3^n(\frac{1}{\sqrt{3}} + 1 - 1)^{2n}}{n^2} = \sum_{n=1}^{+\infty} \frac{3^n(\frac{1}{3})^n}{n^2} = \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

which is a convergent p-series ($p = 2 > 1$). Plugging in $x = -\frac{1}{\sqrt{3}} + 1$ we get:

$$\sum_{n=1}^{+\infty} \frac{3^n(-\frac{1}{\sqrt{3}} + 1 - 1)^{2n}}{n^2} = \sum_{n=1}^{+\infty} \frac{3^n(\frac{1}{3})^n}{n^2} = \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

which, again, is a convergent p-series ($p = 2 > 1$). Thus, the interval of convergence is:

$$\boxed{-\frac{1}{\sqrt{3}} + 1 \le x \le \frac{1}{\sqrt{3}} + 1}$$

Math 181 Final Review Study Guide
Problem 11

12. Find the 5th Taylor polynomial of the function $f(x) = \sin(2x)$ centered at $x = 0$.

Solution: We find the 5th Taylor polynomial using the 5th degree Maclaurin polynomial for $\sin(x)$:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

and replacing x with $2x$:

$$T_5(x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!}$$